On the Theory of Optimal Sensor Placement

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An optimal sensor placement is defined as a sensor configuration that achieves the minimum capital cost while observing prespecified performance criteria. Previous formulations of this problem have resulted in the definition of a mixed-integer nonlinear program (MINLP) with dimensions dependent on the value of the integer decision variables. The main contribution of this work is an equivalent reformulation of the design problem such that the dimension of the NLP is independent of all decision variables. Additionally, the traditional sensor-placement problem, based on static process conditions, is extended to linear dynamic processes. The final contribution is the exact conversion of the general NLP into a convex program through the use of linear matrix inequalities. The aggregation of these results show that the sensor-placement problem can be solved globally and efficiently using standard interior-point and branch-and-bound search algorithms.

Background

It is well known that the performance of any control or monitoring system is strongly dependent on the available process measurements. In fact, the absence of sufficient measurements can significantly complicate the design of a feedback control loop. In an effort to quantify these issues, the steady-state data reconciliation and coaptation algorithm have been developed (Kuehn and Davison, 1961; Madron and Vaverka, 1992; Swartz, 1989; Sanchez and Romagnoli, 1996; Crowe et al., 1983). Of particular importance, the concepts of observability and redundancy in process flow networks were introduced. Observability pertains to the ability to infer anything about a particular process variable based on the given measurements and process configuration, and redundancy is an indicator of the validity of those inferences. Thus, assuming a data-reconciliation algorithm is used, the designer of a control or monitoring system can determine which process variables can be estimated as well as the precision of these estimates.

Turning this problem around, it is often the case that the needs of the control and monitoring systems will drive the sensor-placement activity. As a result, many researchers have sought to develop systematic sensor-placement strategies to meet prespecified performance criteria. One of the earliest such efforts, by Vaclavek and Loucka (1976), sought to achieve the observability requirements of a multicomponent flow network. This type of design criterion is similar to the problem of classification of process variables. The classification procedure uses the process flow diagram to determine which measured variables are redundant and which unmeasured variables are unobservable. An extension of the classification-based design strategy was put forth by Ali and Narasimhan (1993, 1995), wherein they maximize the reliability of the network given the sensor failure rates (they define reliability as the probability of a variable being observable). Madron and Veverka (1992) suggest two optimization problems, one to minimize the difficulty of implementation through a prioritization of measurements from easiest to hardest, and another to maximize the overall precision of the reconciliation. In both cases the performance condition that had to be satisfied was that all process variables of interest (a prespecified set) must be observable. A solution method based on multiple Gauss–Jordan eliminations of the linear mass-balance equations was proposed to find locally optimal solutions to these problems. In Meyer et al. (1994) the authors presented a branch-and-bound-type algorithm to efficiently find the global solutions to the problems presented by Madron and Veverka (1992).

The main difficulty with their procedures is that they consider the property of observability as a true–false property. In reality, the distinction is much more subjective. This is best illustrated by the comments of Madron and Veverka (1992) "... solutions ... which are theoretically observable but with

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unacceptably low precision." With this difficulty in mind, Kretsovalis and Mah (1987) proposed a sensor-placement strategy based on the precision of the reconciled variables. The work by Bagajewicz (1997) extended this idea with the formulation of a minimum capital cost optimization problem subject to reconciliation precision bounds. The formulation also included performance specifications with regard to gross error detectability, resilience to undetected gross errors, as well as precision bounds during sensor fault situations (residual precision). The optimization problem formulated in Bagajewicz (1997) turned out to be a mixed-integer nonlinear program (MINLP). The main difficulty with that formulation was that the dimensions of the reconciliation problem were a function of the integer-constrained decision variables, thus prohibiting the use of traditional branch-and-bound techniques for its solution.

In the next section, we present alternative formulations of the steady-state and dynamic data-reconciliation problems that are especially amiable to the sensor-placement design problem. Then, the performance indicators of precision, gross error detectability, resiliency, residual precision, and reliability are restated in the context of the new formulations. In the fourth and fifth sections, we formally state the optimization problem to be used for design and illustrate how the nonlinear performance conditions can be converted into convex LMI conditions. Finally, examples are presented to illustrate the design procedure.

Reconciliation Problem Formulations

The typical formulation of the linear data-reconciliation problem is as follows (Kuehn and Davidson, 1961). Consider a measurement of the ith process variable, $y_m^{(i)}$. This measurement is related to the actual value of that process variable, $x_m^{(i)}$, through the measurement equation: $y_m^{(i)} = x_m^{(i)} + v^{(i)}$, where $v^{(i)}$ is a zero-mean, normally distributed random variable with variance σ_i^2. If we arrange the measured variables into a column vector, then the measurement equation becomes $y_m = x_m + v$, where the covariance matrix of the vector v is given by the matrix $\Sigma_v (\hat{=} E[vv^T])$. Arranging the remaining process variables, the unmeasured variables, into a column vector, x_u, we find that the network defined mass-balance equations are expressed as

$$A_1 x_m + A_2 x_u = 0 \tag{1}$$

The steady-state data-reconciliation problem is then stated as: Given the measurement data, y_m, estimate the process variables x_m and x_u such that balance equations (Eq. 1) are satisfied and the appropriate least-square residual is minimized. The problem is usually stated in the form of the following optimization problem:

$$\min_{\hat{x}_m,\, \hat{x}_u} \left\{ (y_m - \hat{x}_m)^T \Sigma_v^{-1} (y_m - \hat{x}_m) \right\}$$

$$\text{s.t.} \quad A_1 \hat{x}_m + A_2 \hat{x}_u = 0$$

The optimal solution to this problem, $\hat{x}_m$ and $\hat{x}_u$, is usually called the reconciled value or estimate. In the absence of inequality constraints, the solution to this problem can be determined analytically, as a function of the measurements y_m (see, for example, Madron, 1992; Bagajewicz, 1997). Since the matrices A_1, A_2 are both a function of the number and location of the sensors within the network, the functional form of this analytic solution will be strongly dependent upon the sensor configuration. The primary objective of the following reformulation will be to remove this explicit distinction between measured and unmeasured process variables. It is important to note the reformulation will give exactly the same reconciled values as the traditional ones.

Assume the measurement equation to be $y = x + v$, where x contains all process variables of interest (that is, both measured and unmeasured), and the distinction between measured and unmeasured variables will be implicitly found in the covariance matrix Σ_v. If a variable is intended to be unmeasured, we will express this fact through an infinite variance value for the respective measurement device. Intuitively this makes sense due to the fact that as the variance of a device increases, the information content of its measurements decreases. Thus, in the limit an infinite variance is equivalent to zero information.

It will be assumed throughout that the measurement error terms (the $v^{(i)}$) are independent with respect to each other. Thus, the covariance matrix of v, Σ_v, will always have a diagonal form. This assumption is rather important to the placement algorithms to follow, as it allows for the inverse of Σ_v to be calculated analytically. As we will see shortly, Σ_v will not actually appear in the formulations; however, Σ_v^{-1} (= diag $\{\sigma_i^{-2}\}$) will be central. This is fortunate since Σ_v will become undefined as some of the $\sigma_i \to \infty$, and while Σ_v^{-1} will lose some of its rank, it will always be defined.

In the case of correlated measurement errors, the uncorrelated formulation will still provide a great deal of information. That is, the predicted performance of the reconciliation algorithm cannot be degraded by the presence of unmodeled correlations. The only potential change is if the correlations are subsequently modeled, and in that case, one will find improved performance. Thus, by assuming independence among measurement errors we are allowing for potential improvements in the estimation algorithm, but never degradations.

In an effort to keep the notation simple, we will subsequently insist that the measurement vector, y, contain one and only one element for every process variable. If no physical equipment exists for the measurement of a particular variable (such as a leak stream), then we can always fix the respective inverse variance at zero. We will address the case of multiple-measurement devices on the same process variable shortly.

Steady-state case

Using Gauss–Jordan elimination, the mass-balance equations (Eq. 1) can be represented as $x_s - Bx_p = 0$, where x_p and x_s are classified as primary and secondary process variables, respectively (which are in general different from measured and unmeasured variables). This can always be done if the matrix $[A_1\ A_2]$ contains a full row-rank submatrix. As a matter of bookkeeping, we note that if there are n_y potential measurement points (that is, n_y total process variables) and n_b balance equations, then the number of secondary variables will equal n_b and the number of primary variables will

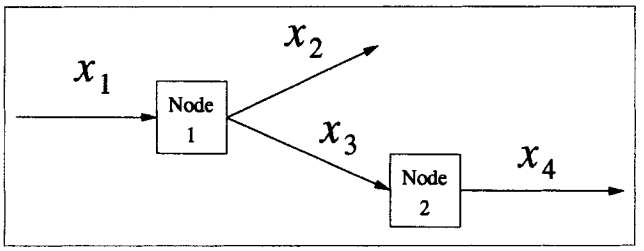

Figure 1. Process flow diagram for Examples 1 and 2.

be the difference, $n_p = n_y - n_b$. Substitution of x_s into the measurement equation yields

$$y = Cx_p + \upsilon \tag{2}$$

where the rows of the C matrix are composed of rows taken from B and the identity matrix.

Example 1a. Consider the flow diagram of Figure 1. If x_2 and x_3 are the primary variables, then the resulting C matrix is

$$C = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \tag{3}$$

The reformulated data-reconciliation problem is now given as

$$Q = \min_{\hat{x}_p} \left\{ \left(y - C\hat{x}_p\right)^T \Sigma_\upsilon^{-1} \left(y - C\hat{x}_p\right) \right\}$$

Since this problem is unconstrained, the solution is found by simply setting the derivative of the objective function to zero. These necessary conditions form the analytic solution:

$$\hat{x}_p = \left(C^T \Sigma_\upsilon^{-1} C\right)^{-1} C^T \Sigma_\upsilon^{-1} y \tag{4}$$

Now reemploying the balance equations in the form of the C matrix we find that the optimal estimates of all the variables are given by $\hat{x} = C\hat{x}_p$.

At first glance, the preceding formulation may appear erroneous due to the need for measured values of potentially unmeasured variables. However, recall that the presence of an unmeasured variable is reflected as a zero value in the matrix Σ_υ^{-1}. From Eq. 4, it is clear that the "measured value" used for an unmeasured stream, $y^{(i)}$ (that is, with $\sigma_i^{-2} = 0$), is inconsequential to the final result. We will see later that the nonsingularity of the matrix $C^T \Sigma_\upsilon^{-1} C$ will determine if all the process variables are observable.

Dynamic case

In the dynamic case, the underlying model is no longer just a set of algebraic equations (i.e., the mass balances), but also includes a set of discrete-time difference equations. Again, the algebraic equations can be expressed as $x_s(k) - Bx_p(k) = 0$, where the argument k is the time index. These equations can be folded into the measurement equation $y(k) = x(k) + \upsilon(k)$, resulting in $y(k) = Cx_p(k) + \upsilon(k)$. Since the secondary variables are completely dependent on the primary variables, we can only have time evolution equations for the primary variables. We propose the following process model for the dynamic data reconciliation problem:

$$x_p(k+1) = Ax_p(k) + Fw(k) \tag{5}$$

$$y(k) = Cx_p(k) + \upsilon(k), \tag{6}$$

where $\upsilon(k)$ and $w(k)$ are random noise sequences with covariance matrices Σ_υ and Σ_w, respectively. Again we have assumed that all process variables (both primary and secondary) are potentially measured and will let Σ_υ^{-1} determine if a given variables is actually measured.

Example 2a. Again, consider the flow diagram of Figure 1 and assume node 2 is a storage tank (possibly with a holdup measurement device). Additionally, a leak stream from the tank is possible, and we would like the ability to estimate such a flow. The mass balances for the system are

$$x^{(1)}(k) = x^{(2)}(k) + x^{(3)}(k) \tag{7}$$

$$m(k+1) = m(k) + x^{(3)}(k) - x^{(4)}(k) - l(k) \tag{8}$$

where $x^{(i)}(k)$ is the total mass flow in stream i between time samples $\Delta t(k-1)$ and Δtk (Δt is the sample rate), $m(k)$ is the tank holdup at time $\Delta t(k-1)$, and $l(k)$ is the total mass leaked between $\Delta t(k-1)$ and Δtk. Let $y(k)$ vector represent measurements taken at time Δtk. If we choose the primary variables as $x_p(k) = [x^{(2)}(k)\ x^{(3)}(k)\ x^{(4)}(k)\ m(k)\ l(k)]^T$, then the C matrix of the measurement equation is

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{9}$$

As suggested in Stanley and Mah (1977) and Chmielewski et al. (2000), we also assume the flow during a given time period will be correlated with the flow during the previous time period [i.e., $x^{(2)}(k+1) = x^{(2)}(k) + w^{(2)}(k)$]. This, along with Eq. 8, gives the following A and F matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{10}$$

However, if we assume that the time period Δt is large (i.e., on the order of hours or days), then the amount of correlation between $x^{(i)}(k)$ and $x^{(i)}(k+1)$ $i = 2, 3, 4$ will actually be small. This assumption is reflected in the values chosen for the diagonal matrix Σ_w. Thus, for large Δt, the first three

diagonal elements of Σ_w^{-1} will approach zero. The last diagonal element of Σ_w, representing the amount of temporal correlation in the leak rate, must be chosen based on physical grounds. Also, note that a measurement for the leak stream is postulated. If there is no way to directly measure the leak rate, then we can easily fix the respective inverse measurement variance at zero.

Although many solution methods have been proposed for dynamic data-reconciliation problems (Bagajewicz and Jiang, 1997; Darouach and Zasadzinski, 1991; Kao et al., 1992; Rollins and Devanathan, 1993; Liebman et al., 1992; Karjala and Himmelblau, 1996), it is safe to assume that the quality of reconciled values will depend more on the sensor configuration than the particular reconciliation algorithm employed. For the analysis to follow, we will assume the employment of the simple but theoretically attractive Kalman filter. The Kalman filter solution, $\hat{x}_p(k)$, is the optimal estimate of the primary process variables, $x_p(k)$. These estimates are given by the recursive equation:

$$\hat{x}_p(k) = A\hat{x}_p(k-1) + P_o C^T \Sigma_v^{-1}\left(y(k) - CA\hat{x}_p(k-1)\right) \quad (11)$$

where P_o must satisfy the algebraic Riccati equation

$$P_o = \left[\left(AP_oA^T + F\Sigma_w F^T\right)^{-1} + C^T\Sigma_v^{-1}C\right]^{-1} \quad (12)$$

The optimal estimates of all the variables are then found through the algebraic equations $\hat{x}(k) = C\hat{x}_p(k)$.

Multiple measurements

A fundamental feature of the data-reconciliation algorithm is to optimally combine measurements resulting from an overspecified sensor configuration (that is, more measurements than the degrees of freedom). In this section we take a closer look at the redundancy case of more than one sensor used to measure the flow of a single stream. Although multiple measurements on a single stream are easily incorporated into most data-reconciliation algorithms, the traditional methods used for incorporation will significantly complicate the sensor-placement algorithm. Thus, in this section we propose a new method of incorporating multiple measurements that is more amiable to the placement activity.

A traditional method of incorporating multiple measurements is to introduce a number of fictitious network nodes. For example, if a single stream has three sensors, then we should introduce two additional nodes, each with one input and one output. From these nodes we have created two additional streams, each with actual flow rates equal to the origional. The result being the introduction of two additional variables and two additional mass-balance equations. Thus, the dimension of the reconciliation problem has been increased, manifesting in the sizes of the C and Σ_v matrices. This change in dimension is, in actuality, not detrimental to the placement algorithm (owing to the reformulation presented above, in which sensors can be removed without impacting the dimensions of the reconciliation problem). The actual problem with this traditional incorporation method is simply the significant increase in dimension that will result. That is, if we would like a formulation that allows each stream the potential of possessing two identical sensors, then the dimension of the reconciliation problem will double. Additionally, if we would like the placement algorithm to choose between n different types of sensors (i.e., with different costs and precisions), then an n-fold increase in the dimensions of C and Σ_v will result.

Now, consider a single process variable $x^{(i)}$ measured by a number of sensors, $y^{(ij)}$, each with a standard deviation σ_{ij}. It is claimed that the following precombination of measurements will yield identical results to that of the traditional incorporation method described in the preceding paragraph (see Appendix A for the proof).

$$y^{(i)} = \sum_j \frac{\sigma_i^2}{\sigma_{ij}^2} y^{(ij)}; \qquad \frac{1}{\sigma_i^2} = \sum_j \frac{1}{\sigma_{ij}^2}$$

Using these definitions, we find that $y^{(i)}$ is a normally distributed random variable with mean $x^{(i)}$ and variance σ_i^2. Thus, we can use the original formulation with our new $y^{(i)}$ and σ_i substituted as usual. The advantage of this method is that no new stream variables or mass balances need to be introduced. Additionally, the sizes of the C and Σ_v matrices are unaffected (that is, they are always $(n_y \times n_p)$ and $(n_y \times n_y)$, respectively).

Performance Indicators

As described in the Background section, there are a number of criteria used to measure the quality of a particular sensor network configuration. In this section, we explicitly define a number of these performance indicators and present the resulting inequality constraints that will be used in the sensor-placement problem formulation.

Estimation precision

The primary statistic used for assessing the performance of a measurement configuration under a reconciliation algorithm is the variance of each of the estimation error terms. The estimation error vector is defined as $\tilde{x} = x - \hat{x}$. This error term measures the precision of the estimation scheme with respect to the actual values being estimated (which are significantly different from the measurement residual $y - C\hat{x}_p$). Unfortunately, the estimation error is unknown to the user and actually can never be calculated. We can, however, calculate its variance theoretically. The error covariance matrix for the steady-state reconciliation problem is

$$S \triangleq E[\tilde{x}\tilde{x}^T] = C\left(C^T\Sigma_v^{-1}C\right)^{-1}C^T \quad (13)$$

The variance associated with each process variable is contained in the appropriate diagonal element of S. Namely, the error variance associated with variable $x^{(i)}$ is

$$s_i \triangleq E\left[\left(x^{(i)} - \hat{x}^{(i)}\right)^2\right] = e_i S e_i^T = c_i\left(C^T\Sigma_v^{-1}C\right)^{-1}c_i^T$$

where e_i and c_i are the ith rows of the identity matrix and C, respectively. In our optimal sensor network problem formulation we will provide upper bounds on the individual error variance terms:

$$c_i\left(C^T\Sigma_v^{-1}C\right)^{-1}c_i^T < \bar{s}_i \tag{14}$$

The performance of the Kalman filter can also be measured with respect to its estimation error: $\tilde{x}(k) \triangleq x(k) - \hat{x}(k) = C[x_p(k) - \hat{x}_p(k)]$. It is well known (Anderson and Moore, 1979) that P_o represents the error covaraince matrix of the primary variables. Thus, the error covariance matrix, S, of all the process variables is given as CP_oC^T. Thus, if we want to bound the error variance of the ith process variable, we must enforce

$$s_i = e_iCP_oC^Te_i^T = c_iP_oc_i^T < \bar{s}_i \tag{15}$$

It is worth noting that in the dynamic case, P_o is analogous to the matrix $(C^T\Sigma_v^{-1}C)^{-1}$ of the steady-state formulation.

Gross-error detectability

Gross-error detection is critical to a sucessful data-reconciliation implementation. The basic approach used in most gross-error detection schemes is to compare the measured data with the expected value of the data based on theoretical considerations. The simplest approach is to analyze the measurement residual, defined as:

$$\tilde{y} \triangleq y - C\hat{x}_p = \left[I - C\left(C^T\Sigma_v^{-1}C\right)^{-1}C^T\Sigma_v^{-1}\right]y$$
$$= \left[I - C\left(C^T\Sigma_v^{-1}C\right)^{-1}C^T\Sigma_v^{-1}\right]v$$

If an element of the vector $\tilde{y}$ is larger than expected, then a gross error is likely present. Another gross-error detection scheme employs the χ^2 test. Using the measurement residuals, one calculates the optimum value of the objective function: $Q(\tilde{y}) = \tilde{y}^T\Sigma_v^{-1}\tilde{y}$. Then, one compares this calculated value with its expected value:

$$E[Q(\tilde{y})] = E\left[v^T\left(I - \Sigma_v^{-1}C\left(C^T\Sigma_v^{-1}C\right)^{-1}C^T\right)\right.$$
$$\left.\Sigma_v^{-1}\left(I - C\left(C^T\Sigma_v^{-1}C\right)^{-1}C^T\Sigma_v^{-1}\right)v\right]$$
$$= \mathrm{Tr}\left\{I - \Sigma_v^{-1}C\left(C^T\Sigma_v^{-1}C\right)^{-1}C^T\right\} = n_y - n_p = n_b$$

where the trace operator, Tr, is defined as the sum of the diagonal elements of a matrix (*Note:* $\mathrm{Tr}\{AB\} = \mathrm{Tr}\{BA\}$ and $\mathrm{Tr}\{A + B\} = \mathrm{Tr}\{A\} + \mathrm{Tr}\{B\}$). Discrepancies between the calculated and expected values are an indication of one or more gross errors being present somewhere in the measurements.

From the preceding two examples we see that the gain from a gross error to the measurement residual is generally a good indicator of detectability. Neglecting the influence of the noise term, consider a measurement $y^{(i)}$ that has been corrupted by a gross error $d^{(i)}$: $y^{(i)} = c_ix_p + d^{(i)}$. We find the impact of the gross error on its measurement residual to be

$$\tilde{y}^{(i)} = \left[1 - c_i\left(C^T\Sigma_v^{-1}C\right)^{-1}c_i^T\sigma_i^{-2}\right]d^{(i)} \tag{16}$$

If multiple-measurement devices are present on the same process variable, then the gross error will first be attenuated by the optimal combination of the duplicate measurements: $y^{(i)} = c_ix_p + (\sigma_i^2/\sigma_{ij}^2)d^{(ij)}$. Recall that σ_{ij}^2 is defined as the variance of the jth sensor at network location i, and not the ijth element of the Σ_v matrix. Now we find the impact of a single gross error, $d^{(ij)}$, to be

$$\tilde{y}^{(ij)} = \left[1 - c_i\left(C^T\Sigma_v^{-1}C\right)^{-1}c_i^T\sigma_{ij}^{-2}\right]d^{(ij)} \tag{17}$$

Using Eq. 17 we define the gross error to residual gain for sensor ij as $k_{ij} = 1 - s_i\sigma_{ij}^{-2}$. Since the error variance of an estimate is always less than or equal to the variance of the related measurement ($s_i \le \sigma_{ij}^2$), it is clear that k_{ij} must be less than or equal to 1. With regard to our sensor network placement problem, we will be interested in providing a lower bound on this gain:

$$k_{ij} = 1 - s_i\sigma_{ij}^{-2} > \bar{k}_{ij}. \tag{18}$$

The interpretation of this inequality is that if a single gross error $d^{(ij)}$ is present (and $v = 0$), then the gross error will appear in the residual $\tilde{y}^{(ij)}$ with a value of at least $\bar{k}_{ij}d^{(ij)}$. We also note that if there is no ij measurement device, then the condition will automatically be satisfied, assuming the absence of the device does not make the network unobservable (that is, $\lim_{\sigma_{ij}\to\infty} s_i < \infty$). In the dynamic case, we only need to replace s_i with its dynamic analog $c_iP_oc_i^T$. Inequality 18 can also be shown to be of the same form as inequality 18 of Bagajewicz (1997). That is, one can rearrange the Bagajewicz inequality into the form just given and conclude that $\bar{k}_{ij}$ is analogus to $(\delta_{i,\gamma}/n_i^D)^2$, where $\delta_{i,\gamma}$ and n_i^D are design parameters defined in Bagajewicz (1997) and are constants to the search algorithm. Thus, we conclude that although the derivation of the two gross-error detectability criteria are different, the resulting inequalities are capable of producing identical results.

Resilience

Another desirable property of a network configuration is the resilience of the estimates with respect to undetected gross errors. One criterion of resiliency is the ability of the reconciliation to attenuate undetected gross errors from the estimation error, $\tilde{x}$. In the presence of an arbitrary gross-error vector, d, the measurement equation is given as $y = Cx_p + v + d$. Assuming that a gross-error detection scheme is being employed, it is reasonable to assume that gross errors with an absolute value greater than some multiple of the standard deviation of the measurement will be detected. Based on this assumption, we normalize the magnitude of each gross error by the standard deviation of the respective measurement. To this end we define: $\delta = \Sigma_v^{-1/2}d$. The esti-

mation error as a function of δ (assuming $v = 0$) is

$$\tilde{x} = H\delta; \quad H = -C\left(C^T\Sigma_v^{-1}C\right)^{-1}C^T\Sigma_v^{-1/2}$$

The impact of the gross-error vector on each element of the estimation-error vector is given by the rows of H, namely, $\tilde{x}^{(i)} = h_i\delta$, where h_i is the ith row of H. Since we do not know the actual direction of the normalized gross error δ, we take a worst-case view of the situation, and determine the maximum possible error over all directions of δ:

$$g_i \triangleq \max_{\|\delta\| \le 1} \{|\tilde{x}^{(i)}|\} = \max_{\|\delta\| \le 1} \{|h_i\delta|\} \tag{19}$$

Now we have some freedom in choosing the norm on δ. The natural choice would be the ∞-norm, $\|\delta\|_\infty = \max_i|\delta^{(i)}|$, which results in $g_i = \Sigma_j|h_{ij}|$, where h_{ij} are the elements of H at row i and column j. However, if we choose the 2-norm, $\|\delta\|_2^2 = \Sigma_i|\delta^{(i)}|^2$, then g_i is found to be $g_i = h_ih_i^T = c_i(C^T\Sigma_v^{-1}C)^{-1}c_i^T = s_i$. Thus, providing a bound in the 2-norm version of g_i has a similar effect as bounding the error variance. It is noted that the preceding definitions of resilience are distinctly different from the definition presented in Bagajewicz (1997).

Residual precision

When a gross error is detected, the usual response is to remove the faulty measurement from the reconciliation algorithm. Residual precision is defined as the estimation-error variance evaluated with the ijth measurement device removed:

$$S^{(i,j)} \triangleq S|_{\sigma_{ij}\to\infty}$$

We can now provide explicit bounds on the residual precision in the form of $c_kS^{(i,j)}c_k^T < \bar{s}_{ijk}$. In addition, this definition can easily be extended to the removal of pairs and triples of measurements. While these definitions are conceptually simple, their implementation may be cumbersome due to the large number of constraints generated by the many i, j, k combinations possible. Instead of enforcing every combination of the condition, a more reasonable approach is to guarantee residual precision only with respect to sensor faults local to the variable of interest.

Reliability

While the cost vs. precision ratio of a sensor is a key factor in choosing sensors for a network, an additional consideration is the reliability of the sensors. One of the drawbacks of the residual precision notion is that it does not incorporate the sensor failure rates into the section process. In the context of data reconciliation, the notion of reliability is defined as the probability of a variable being observable (Ali and Narasimhan, 1993). Clearly, this probability is a function of the failure rates of the sensors. A more quantitative definition is $Pr[s_i < \bar{s}_i]$, the probability that the estimation precision will satisfy prespecified bounds. In terms of a design criterion we propose the following:

$$Pr[s_i < \bar{s}_i] \triangleq r_i > \bar{r}_i \tag{20}$$

If Ω is the sample space of independent failure scenarios, then the probability of interest is calculated as the sum over all failure scenarios that lead to a larger precision than the threshold.

$$r_i = \sum_{\omega_k \in \Omega} Pr(\omega_k)I(\omega_k)$$

$$\text{where} \quad I(\omega_k) = \begin{cases} 1 & \text{if} \quad s_i(\omega_k) < \bar{s}_i \\ 0 & \text{if} \quad s_i(\omega_k) \ge \bar{s}_i \end{cases} \tag{21}$$

Given the failure rates of the individual sensors, we can easily precalculate $Pr(\omega_k)$ for every scenario. A minor difficulty with this criterion deals with the large number of possible failure scenarios (if the maximum number of sensors possible in a network is n, then Ω will contain at most 2^n elements). Again, it is easily seen that sensor failures far from a process variable of interest will affect the estimate precision only marginally if at all. If we assume these distant failures do not affect the precision at all, then a significant reduction in the number of summations is achieved. A more difficult problem involves the severe nonlinearity of the $I(\omega_k)$ function. In a later section, we suggest a smooth approximation of this function, thereby providing a smooth approximation of r_i.

Optimal Sensor Nework Problem Formulation

The optimal sensor network problem is formulated as follows. Let q_{ij} be a set of zero-one decision variables corresponding to the placement of sensor type j at network location i. Then our objective is to minimize the capital cost, Z, of implementing the given network configuration:

$$Z = \sum_i \sum_j z_jq_{ij} \tag{22}$$

where z_j is the cost of each sensor type. This minimization can now be subject to any of the performance criteria described in the previous section. Each of these performance inequalities is a function of the measurement covariance matrix Σ_v, which is defined to be a diagonal matrix with diagonal elements σ_i^2 defined by

$$\frac{1}{\sigma_i^2} = \sum_j \frac{q_{ij}}{\bar{\sigma}_{ij}^2}$$

The interpretation of the q_{ij} in Σ_v are as follows. If $q_{ij} = 1$, then sensor type j is present at location i and has a standard deviation of $\bar{\sigma}_{ij}$ ($\bar{\sigma}_{ij}$ is a fixed parameter of the optimization problem). If $q_{ij} = 0$, then the standard deviation of sensor ij is infinite, which is equivalent to not having that sensor present.

The most important feature of the preceding formulation is that the inverse of Σ_v is linear with respect to the q_{ij} decision variables. However, the performance conditions are still nonlinear. These nonlinearities result in the optimization problem being an MINLP. The usual method used to solve

an MINLP is to employ a branch-and-bound algorithm. Proper implementation of this algorithm requires the ability to find the global optimum of the NLP resulting from a relaxation of the integer constraints. In the next section, we will convert the general nonlinear constraints into equivalent convex feasibility conditions. Through these transformations we will have converted the diffcult-to-solve NLPs into significantly more manageable convex programs, which allows us to efficiently find the global solution to the integer constraint relaxed subproblems. Before presenting these transformations, we must address the more subtle issue of allowing certain variables to be unobservable.

Unobservable variables

A point crucial to our problem formulation is the handling of unobservable variables. In the reformulated reconciliation problem, we have implicitly assumed that all the variables are somehow observable. If one of our variables was in fact unobservable, then this would be reflected in the singularity of the matrix $(C^T\Sigma_v^{-1}C)$. As our sensor-placement design method searches for the "best" Σ_v, any choice that results in an unobservable variable, anywhere in the network, will be found to be infeasible due to the singularity. However, there may be cases in which a configuration that contains an unobservable variable is perfectly feasible. This situation is best illustrated by an example. Consider a single node with two input streams and only one output stream. If the only performance requirement is a bound on the error variance of the output stream, then a number of feasible configurations may exist. The first is to have sufficiently precise sensors on at least two of the three streams. Clearly, all variables will be observable in this configuration. Another configuration is to just have a sensor on the output stream. If that sensor is sufficiently precise, then clearly the performance requirement will be met. However, due to the unobservability of the inlet streams, the reconciliation problem will be ill-defined and the branch-and-bound algorithm will consider this configuration to be infeasible. If the problem was formulated such that bounds were placed on the estimation error of the inlet streams, then the infeasibility of the single-sensor configuration would be warranted. However, since the designer has expressed no interest in the inlet streams, we must allow this configuration to be feasible. In Bagajewicz (1997), the author suggests removing all unobservable variables from the network before calculating the error variance, but admits that this approach precludes the use of traditional branch-and-bound algorithms.

A simple method to solve this problem is to guarantee the presence of a very poor sensor at each stream. This will manifest as a very small constant value in each of the diagonal elements of Σ_v^{-1}. These small positive constants will guarantee that Σ_v^{-1} is just barely nonsingular, in turn implying that $C^T\Sigma_v^{-1}C$ is barely nonsingular (since the null space of C is empty). A side effect of these fictitious sensors is that the error variance will be slightly reduced. If we want to minimize this effect, we could apply the fictitious sensors only to streams that do not already have performance bounds on their error variances. Alternatively, the inclusion of poor sensors can be decided automatically be introducing decision variables, q_{i0}. These sensors should have a low cost (but large cost to precision ratio). We can ensure that these fictitious devices are only used to achieve observability by enforcing the following linear constraints: $q_{i0} + q_{ij} \le 1$ for each j. Since the q's are zero-one variables, these constraints only allow the fictitious sensor to be used if no other sensors are being used. The proposed convexification methods to follow will require the assumption that $C^T\Sigma_v^{-1}C$ is strictly positive definite, as opposed to just positive semidefinite. The preceding discussion shows that we can easily guarantee this condition.

Convex Performance Specifications

In this section, we illustrate how the previously defined nonlinear performance indicators can be converted to convex inequality constraints. In all cases, these convex constraints will take the form of linear matrix inequalities (LMIs). For an introduction to LMI technology the reader is referred to the following: Balakrishnan et al., 2000; Skelton et al., 1999; and VanAntwerp and Braatz, 2000. Also, the Matlab LMI Control Toolbox manual (1995) contains an informative introduction section. From this literature, it is made clear the newly formulated LMI constrained optimization problem will become convex. Thus we are guaranteed that the solutions to the integer constraint relaxed subproblems of the branch-and-bound algorithm will be globally optimal. This global optimality is crucial to the branch-and-bound method, as the method may produce suboptimal solutions without such a guarantee.

Precision constraints

The first set of performance specifications we will address are the bounds on the estimation-error variances. Using the Schur Complement Theorem (see fact A0 of Appendix B), we can easily show the equivalence of inequality 14 with the following matrix inequality:

$$\begin{bmatrix} \bar{s}_i & c_i \\ c_i^T & C^T\Sigma_v^{-1}C \end{bmatrix} > 0 \qquad (23)$$

Recall that Σ_v^{-1} is a linear function of the decision variables q_{ij}. Thus, Inequality 23 is an LMI condition.

In the dynamic case, the situation is more complicated due to the required satisfaction of Eq. 12. The following theorem addresses the central issues pertaining to the dynamic case (the proof can be found in Appendix B).

Theorem 1. Let $\Sigma_v > 0$, $\Sigma_w > 0$, A^{-1} exist, and the pair $(A, F\Sigma_w^{1/2})$ be stabilizable. Then there exists $P_o > 0$ s.t. $\mathrm{Tr}\{U^TP_oU\} < \gamma^2$ and Eq. 12 is satisfied if and only if there exists $X > 0$ s.t. $\mathrm{Tr}\{U^T(A^TXA)^{-1}U\} < \gamma^2$ and

$$\begin{bmatrix} X - A^TXA + C^T\Sigma_v^{-1}C & XF \\ F^TX & F^TXF + \Sigma_w^{-1} \end{bmatrix} > 0 \qquad (24)$$

Using Theorem 1 and the Schur Complement Theorem, the equivalence of Inequality 15 and the following set of LMIs is

transparent ($U = c_i^T$):

$$\begin{bmatrix} \bar{s}_i & c_i \\ c_i^T & A^T X A \end{bmatrix} > 0 \quad \text{and}$$

$$\begin{bmatrix} X - A^T X A + C^T \Sigma_v^{-1} C & XF \\ F^T X & F^T X F + \Sigma_w^{-1} \end{bmatrix} > 0 \quad (25)$$

Since the resilience and residual precision conditions are of the same form as the precision constraints, similar methods can be employed for their conversion to LMIs.

Detectability constraints

Our constraint ensuring gross-error detectability is summed up in Inequality 18. Rearranging Inequality 18 and employing $\sigma_{ij}^{-2} = q_{ij}\bar{\sigma}_{ij}^{-2}$, we find

$$s_i = c_i \left(C^T \Sigma_v^{-1} C\right)^{-1} c_i^T < \left(1 - \bar{k}_{ij}\right) \frac{\bar{\sigma}_{ij}^2}{q_{ij}} \quad (26)$$

The following lemma is central to converting this inequality to a convex condition.

Lemma 1. Let a be a positive constant. If there exists $q \in (0,1)$ such that $s(q) < a/q$, then there exists $q \in (0,1)$ such that $s(q) < a/q^2$.

Proof. $(a/q) < (a/q^2)$, $\forall q \in (0,1)$.

Thus, if we replace Eq. 26 with $s_i < (1 - \bar{k}_{ij})(\bar{\sigma}_{ij}^2/q_{ij}^2)$, we will have enlarged the feasible region of the NLP problem with the integer constraints relaxed. However, since the actual q variables are limited to being zero-one integers, there is no change in the actual feasible region. Employing the Schur Complement Theorem, we find Eq. 26 to be equivalent to

$$\begin{bmatrix} \left(1 - \bar{k}_{ij}\right)\bar{\sigma}_{ij}^2 & q_{ij} c_i \\ q_{ij} c_i^T & C^T \Sigma_v^{-1} C \end{bmatrix} > 0 \quad (27)$$

This condition agrees with our origional observation that the condition is satisfied if there is no sensor at location ij (assuming $C^T \Sigma_v^{-1} C > 0$, when $q_{ij} = 0$).

Reliability constraints

As discussed earlier, the main difficulty with enforcing Eq. 20 is the calculation of the nonlinear indicator function $I(\omega_k)$. The following development provides an approximation of $I(\omega_k)$. Recall that $I(\omega_k)$ is defined to be 1 if the fault configuration ω_k leads to $s_i(\omega_k) < \bar{s}_i$, and equals 0 otherwise. Thus, if we define $0 < \epsilon_{ik} \leq 1$ as a slack variable in the condition

$$s_i(\omega_k) < \frac{\bar{s}_i}{\epsilon_{ik}} \Leftrightarrow s_i(\omega_k)\epsilon_{ik}^2 < \bar{s}_i \epsilon_{ik} \quad (28)$$

then the condition can always be met as long as $s_i(\omega_k) < \infty$. Clearly, ϵ_{ik} approximates $I(\omega_k)$ in the sense that if $s_i(\omega_k) < \bar{s}_i$ is feasible, then ϵ_{ik} can equal 1, but if not, then ϵ_{ik} must be less than 1. Employing the Schur Complement Theorem on Eq. 28, we find

$$\begin{bmatrix} \bar{s}_i \epsilon_{ik} & \epsilon_{ik} c_i \\ c_i^T \epsilon_{ik} & C^T \Sigma_v(\omega_k)^{-1} C \end{bmatrix} > 0 \quad (29)$$

However, if $s_i(\omega_k)$ is sufficiently large, then the small ϵ_{ik} required to satisfy Eq. 28 will make the matrix in Eq. 29 appear singular. To fix this problem, we add a small constant, $\bar{s}_o$, to the one-one element of Eq. 29. Thus, the following LMIs will approximate the reliability condition expression in Eq. 20

$$\sum_{\omega_k \in \Omega} Pr(\omega_k)\epsilon_{ik} > \bar{r}_i \quad \text{and}$$

$$\begin{bmatrix} \bar{s}_o + \bar{s}_i \epsilon_{ik} & \epsilon_{ik} c_i \\ c_i^T \epsilon_{ik} & C^T \Sigma_v(\omega_k)^{-1} C \end{bmatrix} > 0 \quad (30)$$

Design Examples

In this section, we present a number of examples to illustrate the design methodology. As described earlier, the sensor-placement problem can now be solved using the branch-and-bound algorithm with the usual linear programs (LPs) replaced by LMI-constrained minimization problems. We solved the LMI-constrained subproblems using the "LMI Control Toolbox" for use with Matlab. A detailed description of these standard search algorithms can be found in Edgar and Himmelblau (1988) for branch-and-bound, and Nesterov and Nemirovsky (1994) or Vandenberghe and Balakrishnan (2000) for the LMI subproblems.

The implementation was relatively straightforward with the exception of a few quirks regarding the LMI Control Toolbox. In particular, the toolbox does not provide a simple method to implement equality constraints, even on scalar variables. Since equality constraints are fundamental to the branch-and-bound algorithm, we had to find an accurate way to implement them. The basic approach was to simply substitute the equality constraints into LMI conditions and then redefine the subproblem with the remaining q_{ij} as decision variables. The following expansion of $C^T \Sigma_v^{-1} C$ proved useful in our redefinition of the subproblems:

$$C^T \Sigma_v^{-1} C = \sum_i \sum_j q_{ij} \frac{c_i^T c_i}{\bar{\sigma}_{ij}^2}$$

Another difficulty encountered, which also stemmed from the toolbox's ability only to enforce strict inequalities, was the determination of active constraints (using this information, we can significantly improve the efficiency of the branch-and-bound algorithm). To circumvent this problem, we added a postprocessing algorithm to determine if postulated active constraints were indeed feasible as well as provided lower cost.

Example 1b (Steady-State Process with Precision Constraints Only): Here we continue Example 1a (that is, with a flow diagram as shown in Figure 1 and a C matrix given by Eq. 3).

Table 1. Measurement Precisions vs. Sensor Type and Stream Number

Sensor Type (j) (%)	Precision in Stream 1 $(\bar{\sigma}_{1j})$	Precision in Stream 2 $(\bar{\sigma}_{2j})$	Precision in Stream 3 $(\bar{\sigma}_{3j})$	Precision in Stream 4 $(\bar{\sigma}_{4j})$	Capital Cost (z_j)
1	1.5010	0.5230	0.9780	0.9780	\$2,500
2	3.0020	1.0460	1.9560	1.9560	\$1,500
3	4.5030	1.5690	2.9340	2.9340	\$800

Table 2. Measurement Precisions vs. Sensor Type and Stream Number

Sensor Type (j) (%)	Precision in Stream 1 $(\bar{\sigma}_{1j})$	Precision in Stream 2 $(\bar{\sigma}_{2j})$	Precision in Stream 3 $(\bar{\sigma}_{3j})$	Precision in Stream 4 $(\bar{\sigma}_{4j})$	Capital Cost (z_j)
1	28.5	9.9	18.5	18.5	\$2,500
2	57.0	19.8	37.1	37.1	\$1,500
3	85.4	29.8	55.7	55.7	\$800

We assume the nominal flow rates and sensor options to be as in example 1 of Bagajewicz (1997) (i.e., $x = [150.1, 52.3, 97.8, 97.8]$ and 1%, 2%, or 3% relative errors). The resulting measurement precisions (in kg/min) and respective capital costs are presented in Table 1. The performance specifications used in Bagajewicz (1997) require that the relative error of the reconciled variables at streams 1 and 4 be less than 1.5% and 2%, respectively. These specifications translate into $\bar{s}_1 = 5.0693$ kg^2/min^2 and $\bar{s}_4 = 3.8259$ kg^2/min^2. The set of optimal solutions was found to be the same as in Bagajewicz (1997): a 2% sensor at stream 2 and a 2% sensor at either stream 3 or stream 4, with a cost of \$3,000. Of the 2^{12} possible sensor configurations, the branch-and-bound algorithm found the first of these solutions on its 36th iteration and was able to finally guarantee its optimality after the 205th.

Example 1c (Steady-State Process with Precision and Detectability Constraints). Using the design parameters of Example 1b, we illustrate the impact of constraints on gross-error detectability. First, we set $\bar{k}_{ij} = 0.01$ for all i and j. In this case, the optimal set of solutions was found to be a 3% sensor on stream 1, a 3% sensor on stream 2, and a 2% sensor on either steam 3 or stream 4, with a cost of \$3,100. Note that $k_{ij} = 0.01$ is equivalent to allowing only 1% of all gross errors to pass through, which is virtually undetectable; yet, it does make the network minimally redundant. When the gross-error detectability was set to $\bar{k}_{ij} = 0.1$, the minimum cost increased to \$3,800, which was achieved by the following configurations: a 2% sensor on stream 1, a 3% sensor on stream 2, and a 2% sensor on either strean 3 or stream 4.

Example 2b (Dynamic Process with Precision Constraints). Here we continue Example 2a (i.e., with a flow diagram as shown in Figure 1, and C, A, and F matrices given by Eqs. 9 and 10). If we choose a sample time Δt to be one day, then based on the discussion in Example 2a, a reasonable choice for Σ_w^{-1} is diag{0, 0, 0, $1/(2.15)^2$}. The nonzero term indicates that the total mass lost per day will change by approximately ± 2.15 kg/day. Again, we assume there are three measurement-device options for each stream (1%, 2%, and 3% relative error with costs of \$2,500, \$1,500, and \$800, respectively), and the nominal stream flow rates are $x = [150.1, 52.3, 97.8, 97.8]$ kg/min $= [216144, 75312, 140832, 140832]$ kg/day. Using the error analysis methods presented in Chmielewski et al. (2000), the resulting measurement precisions are presented in Table 2 (in kg). In addition, there were two options for the tank holdup measurement with standard deviations of 16 and 64 kg and a capital cost of \$2,600 and \$500, respectively. Finally, the performance requirements were specified to be $\bar{s}_1 = 1824.9$ kg^2, $\bar{s}_4 = 1377.3$ kg^2, and $\bar{s}_6 = 100.0$ kg^2. For this set of parameters, the optimal solution was found to be a 3% sensor on stream 2, a 1% sensor on stream 3, a 2% sensor on stream 4, and the 64-kg holdup measurement, with a cost of \$5,300.

Conclusion

In this article, we considered the design of data-reconciliation enhanced sensor networks. The current state-of-the-art design methodologies were reviewed while highlighting some of their limitations. We proposed an alternative but equivalent formulation of the reconciliation problem, with the main advantage being the ability to exchange measured and unmeasured variables without having to change the structure or dimension of the problem. Additionally, the nonlinear equality and inequality constraints of the resulting optimal sensor placement problem were shown to be exactly equivalent to a set of convex inequality constraints in the form of LMI conditions. In sum, the main contribution of this work has been to reformulate the optimal sensor placement problem in such a way that its global solution can be efficiently determined using established search algorithms. Using the proposed method, one can quickly and easily calculate the minimum cost associated with the imposition of physically meaningful performance criteria. Thus, bringing the designers of control and monitoring systems a step closer to the establishment of rigorous cost-to-performance trade-off curves.

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Notation

A_1, A_2, B, C = matrices defining network dynamics
A, F = matrices defining mass-balance equations
c_i, e_i = ith row of the C and identity matrices
diag$\{\cdot\}$ = diagonal matrix with elements given by the vector argument
$d^{(i)}, d^{(ij)}$ = gross error at measurement point i, sensor type j
$E[\cdot]$ = expectation operator
g_i = gain from normalized gross error to estimation error of variable i
δ = normalized vector of gross errors
h_i, h_{ij} = ith row and ijth element of matrix H
H = matrix relating normalized gross error to estimation error
$k, \Delta t$ = time index and sample time
k_{ij} = gain from gross error to measurement residual
$l(k)$ = total mass leaked between times $\Delta t(k-1)$ and Δtk
$m(k)$ = tank holdup at time $\Delta t(k-1)$
n_y, n_b, n_p = number of measurements, balance equations, and primary variables
P_o = covariance matrix of primary variable estimation errors

$Pr[\omega_k], I(\omega_k)$ = probability and indicator function for failure scenario ω_k

q_{ij} = decision variable for sensor ij

Q = value of the reconciliation objective function

r_i = probability s_i will satisfy bound $\bar{s}_i$ (reliability)

s_i = ith diagonal element of S matrix

$\bar{s}_i, \bar{k}_{ij}, \bar{r}_i$ = upper bound design specifications

S = covariance matrix of estimation errors for all variables

$*^T$ = transpose operator

$\mathrm{Tr}\{\cdot\}$ = trace operator (sum of diagonal elements of square matrix)

v = vector of measurement errors

$v(k), w(k)$ = vector of measurement errors and input disturbances at time k

x = vector of actual variable values

x_m, x_u = vector of actual values for measured and unmeasured variables

x_p, x_s = vector of actual values for primary and secondary variables

$\tilde{x}$ = vector of errors between actual and estimated values

$\hat{x}, \hat{x}_p$ = vector of estimated variable values

$\hat{x}_m, \hat{x}_u$ = vector of estimated values for measured and unmeasured variables

y, y_m = vector of measurement realizations

$\tilde{y}$ = vector of errors between measurements and estimates (residual)

$y(k), x(k), \bar{x}(k)$ = vector of measured, actual, and esimated variables at time k

z_j, Z = cost of sensor type j and total cost of optimal sensor network

$\Omega, \omega_k \in \Omega$ = sample space of failure scenarios and scenario number k

$\bar{\sigma}_{ij}$ = standard deviation of sensor type j at location i

$\sigma_i^2, \sigma_{ij}^2$ = variance of the measurement error at location i, sensor type j

Σ_v = covariance matrix of the error vector v

Σ_w = covariance matrix of disturbance vector $w(k)$

Literature Cited

Ali, Y., and S. Narasimhan, "Sensor Network Design for Maximizing Reliability of Linear Processes with Redundant Sensors," *AIChE J.*, **39**, 820 (1993).

Ali, Y., and S. Narasimhan, "Redundant Sensor Network Design for Linear Processes," *AIChE J.*, **41**, 2237 (1995).

Anderson, B., and J. Moore, *Optimal Filtering*, Prentice Hall, Englewood Cliffs, NJ (1979).

Bagajewicz, M., "Design and Retrofit of Sensor Networks in Process Plants," *AIChE J.*, **43**, 2300 (1997).

Bagajewicz, M., and Q. Jiang, "Integral Approach to Plant Linear Dynamic Reconciliation," *AIChE J.*, **43**, 2546 (1997).

Balakrishnan, A. V., *Kalman Filtering Theory*, Optimization Software, New York (1987).

Balakrishnan, V., F. Wang, and L. Vandenberghe, "Applications of Semidefinite Programming in Process Control," *Proc. Amer. Cont. Conf.*, Chicago, p. 3219 (2000).

Chmielewski, D., V. Manousiouthakis, B. Tilton, and B. Felix, "Loss Accounting and Estimation of Leaks and Instrument Biases Using Time-Series Data," *Ind. Eng. Chem. Res.*, **39**, 2336 (2000).

Crowe, C. M., Y. A. Garcia Campos, and A. Hrymak, "Reconciliation of Process Flow Rates by Matrix Projection: I. The Linear Case," *AIChE J.*, **29**, 818 (1983).

Darouach, M., and M. Zasadzinski, "Data Reconciliation in Generalized Linear Dynamic Systems," *AIChE J.*, **37**, 193 (1991).

Edgar, T. F., and D. M. Himmelblau, *Optimization of Chemical Processes*, McGraw-Hill, New York (1988).

Kao, C., A. C. Tamhane, and R. S. H. Mah, "Gross Error Detection in Serially Correlated Process Data. 2. Dynamic Systems," *Ind. Eng. Chem. Res.*, **31**, 254 (1992).

Karjala, T. W., and D. M. Himmelblau, "Dynamic Rectification of Data via Recurrent Neural Nets and the Extended Kalman Filter," *AIChE J.*, **42**, 2225 (1996).

Kretsovalis, A., and R. S. H. Mah, "Effect of Redundancy on Estimation Accuracy in Process Data Reconciliation," *Chem. Eng. Sci.*, **42**, 2115 (1987).

Kuehn, D. R., and H. Davison, "Conputer Control: II. Mathematics of Control," *Chem. Eng. Prog.*, **57**, 44 (1961).

Liebman, M. J., T. F. Edgar, and L. S. Lasdon, "Efficient Data Reconciliation and Estimation for Dynamic Processes Using Nonlinear Programming Techniques," *Computer Chem. Eng.*, **16**, 963 (1992).

LMI Control Toolbox User's Guide, The MathWorks Inc., Natick, MA (1995).

Madron, F., and V. Veverka, "Optimal Selection of Measuring Points in Complex Plant by Linear Methods," *AIChE J.*, **38**, 227 (1992).

Madron, F., *Process Plant Performance, Measurement Data Processing for Optimization and Retrofits*, Ellis Horwood, West Sussex, England (1992).

Meyer, M., J. M. Le Lann, B. Koehret, and M. Enjalbert, "Optimal Selection of Sensor Location on a Complex Plant Using a Graph Oriented Approach," *Comput. Chem. Eng.*, **18**, S535 (1994).

Nesterov, Y., and A. Nemirovsky, *Interior-Point Polynomial Methods in Convex Progamming*, Vol. 13 of *Studies in Applied Mathematics*, SIAM, Philadelphia, PA (1994).

Rollins, D. K., and S. Devanathan, "Unbiased Estimation in Dynamic Data Reconciliation," *AIChE J.*, **39**, 1330 (1993).

Sanchez, M., and J. Romagnoli, "Use of Orthogonal Transformations in Classification/Data Reconciliation," *Comput. Chem. Eng.*, **20**, 483 (1996).

Skelton, R. E., T. Iwasaki, and K. M. Grigoriadis, *A Unified Algebraic Approach to Linear Control Design*, Taylor & Francis, New York (1999).

Stanley, G. M., and R. S. H. Mah, "Estimation of Flows and Temperatures in Process Networks," *AIChE J.*, **23**, 642 (1977).

Swartz, C. L. E., "Data Reconciliation of Generalized Flowsheet Applications," ACS Meeting, Dallas (1989).

Vaclavek, V., and M. Loucka, "Selection of Measurements Necessary to Achieve Multicomponent Mass Balances in Chemical Processes," *Chem. Eng. Sci.*, **31**, 1199 (1976).

VanAntwerp, J. G., and R. B. Braatz, "A Tutorial on Linear and Bilinear Matrix Inequalities," *J. Process Control*, **10**, 363 (2000).

Vandenberghe, L., and V. Balakrishnan, "Algorithms and Software for LMI Problems in Control," *IEEE Control Syst. Mag.*, 89 (1997).

Appendix A

To prove the equivalence of the two multiple-measurement formulations we need only show that the two primary-variable estimates are equal. Consider a reconciliation problem with n_y measurements and n_p primary variables. For the moment we will assume that at most one sensor is applied to each stream. For this case, the following definitions apply

$$y = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(n_y)} \end{bmatrix} \quad C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n_y} \end{bmatrix} \quad \Sigma_v = \begin{bmatrix} \sigma_1^2 & 0 & 0 & 0 \\ 0 & \sigma_2^2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \sigma_{n_y}^2 \end{bmatrix}$$

where c_i are the rows of the C matrix. If we now add a redundant sensor to the measurement $y^{(n_y)}$, the "fictitious node" formulation will result in the following definitions:

$$\tilde{y} = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(n_y)} \\ y^{(n_y+1)} \end{bmatrix} \quad \bar{C} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n_y} \\ c_{n_y} \end{bmatrix}$$

$$\bar{\bar{\Sigma}}_v = \begin{bmatrix} \sigma_1^2 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2^2 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \sigma_{n_y}^2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_{n_y+1}^2 \end{bmatrix}$$

Note that although the dimention of the C matrix has increased by one row, this additional row, c_{n_y+1} is identical to c_{n_y}, due to the redundant nature of sensor $y^{(n_y+1)}$. Now consider the same case of adding a redundant sensor adjacent to $y^{(n_y)}$, but this time employ the new formulation. In this case the following definitions apply:

$$\bar{\bar{y}} = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ \alpha y^{(n_y)} + \beta y^{(n_y+1)} \end{bmatrix}$$

$$\bar{\bar{\bar{\Sigma}}}_v = \begin{bmatrix} \sigma_1^2 & 0 & 0 & 0 \\ 0 & \sigma_2^2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \left(\sigma_{n_y}^{-2} + \sigma_{n_y+1}^{-2}\right)^{-1} \end{bmatrix}$$

and $\bar{\bar{\bar{C}}} = C$, where $\alpha = (\sigma_{n_y}^{-2} + \sigma_{n_y+1}^{-2})^{-1}/\sigma_{n_y}^2$ and $\beta = (\sigma_{n_y}^{-2} + \sigma_{n_y+1}^{-2})^{-1}/\sigma_{n_y+1}^2$. Comparing the two approaches, one can easily verify that

$$\bar{C}^T \bar{\bar{\Sigma}}_v^{-1} \bar{C} = C^T \bar{\bar{\bar{\Sigma}}}_v^{-1} C$$

indicating that the covariance matrices of the primary variable estimation errors are the same in both cases. Additionally, inspection of the terms $\bar{C}^T \bar{\bar{\Sigma}}_v^{-1} \bar{\bar{y}}$ and $\bar{\bar{C}}^T \bar{\bar{\bar{\Sigma}}}_v^{-1} \bar{\bar{\bar{y}}}$ indicate that they too are identical. This along with the preceding conclusion show that the primary variable estimates are the same in each case. A repetition of this procedure for each hardware redundant sensor completes the proof.

Appendix B

The proof of Theorem 1 is given in two parts. The first, Lemma A1, shows that an algebraic Riccati equation (ARE) (which is different in form than Eq. 12) is equivalent to a similar algebraic Riccati inequality (ARI). The second part shows that the two AREs and the two ARIs are equivalent. The following facts are standard results from the theory of linear algebra and are restated here for convenience.

Fact A0 (Schur Complement Theorem). Let M be a symmetric matrix partitioned as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix}$$

Then M is positive definite (>0) if and only if $M_{22} > 0$ and $M_{11} - M_{12} M_{22}^{-1} M_{12}^T > 0$.

Fact A1 (Matrix Inversion Lemma). Assume all required inverses exist and the matrices $A = A^T$, $B = B^T$, C are of appropriate dimension, then $(A^{-1} + CB^{-1}C^T)^{-1} = A - AC(C^TAC + B)^{-1}C^TA$.

Fact A2. Assuming A is stable (all eignvalues of A are strictly within the unit circle) and $Q > 0$, then

$$Y = \sum_{k=0}^{\infty} A^k Q A^{Tk}$$

satisfies $Y = AYA^T + Q$.

Fact A3. There exists $Y > 0$ such that $AYA^T < Y$ if and only if A is stable.

Fact A4. If A is stable and $Y = Y^T$ satisfies $AYA^T < Y$, then $Y > 0$.

Lemma A1. Let $\Sigma_v > 0$, $\Sigma_w > 0$, A^{-1} exist and the pair $(A, F\Sigma_w^{1/2})$ be stabilizable. Then there exists $Y_o > 0$ s.t. $\mathrm{Tr}\{U^T[Y_o^{-1} + C^T\Sigma_v^{-1}C]^{-1}U\} < \gamma^2$ and

$$-Y_o + AY_oA^T - AY_oC^T\left(\Sigma_v + CY_oC^T\right)^{-1}CY_oA^T + F\Sigma_wF^T = 0 \quad \text{(B1)}$$

if and only if there exists $Y > 0$ s.t. $\mathrm{Tr}\{U^T[Y^{-1} + C^T\Sigma_v^{-1}C]^{-1}U\} < \gamma^2$ and

$$-Y + AYA^T - AYC^T\left(\Sigma_v + CYC^T\right)^{-1}CYA^T + F\Sigma_wF^T < 0 \quad \text{(B2)}$$

Proof. Only if part: Define $Y > 0$ as follows

$$Y \triangleq \sum_{k=0}^{\infty} (A - M_oC)^k \Lambda \left(A^T - C^TM_o^T\right)^k \quad \text{(B3)}$$

where $M_o \triangleq AY_oC^T(CY_oC^T + \Sigma_v)^{-1}$ and $\Lambda \triangleq F\Sigma_wF^T + M_o\Sigma_vM_o^T + \epsilon^2 I$. Equation B1 and the fact that $(A, F\Sigma_w^{1/2})$ are a stabilizable pair guarantee that $A - M_oC$ is stable (Balakrishnan, 1987), in turn guaranteeing that Y will exist. Thus, Y satisfies

$$\begin{aligned} 0 &= -Y + (A - M_oC)Y(A - M_oC)^T + \Lambda \\ &= -Y + (A - MC)Y(A - MC)^T + F\Sigma_wF^T + M\Sigma_vM^T \\ &\quad + \epsilon^2 I + (M - M_o)(CYC^T + \Sigma_v)(M - M_o)^T \end{aligned}$$

where $M \triangleq AYC^T(CYC^T + \Sigma_v)^{-1}$. Since $\epsilon^2 I + (M - M_o)(CYC^T + \Sigma_v)(M - M_o)^T > 0$, we conclude that Y satisfies Inequality B2. Using the matrix inversion lemma on Inequality B2 we find that

$$-Y + F\Sigma_wF^T + A\left[Y^{-1} + C^T\Sigma_v^{-1}C\right]^{-1}A^T < 0$$

Additionally, from the definition of Y we conclude that $Y = Y_o + \epsilon^2\tilde{Y}$, where $\tilde{Y} > 0$. These facts along with Eq. B1 allow

us to conclude that

$$\left[Y^{-1}+C^T\Sigma_v^{-1}C\right]^{-1}<\left[Y_o^{-1}+C^T\Sigma_v^{-1}C\right]^{-1}+\epsilon^2A^{-1}\tilde{Y}(A^T)^{-1}$$

which implies

$$\mathrm{Tr}\left\{U^T\left[Y^{-1}+C^T\Sigma_v^{-1}C\right]^{-1}U\right\}<\gamma^2+\epsilon^2\mathrm{Tr}\left\{U^TA^{-1}\tilde{Y}(A^T)^{-1}U\right\}$$

Thus, for small enough $\epsilon>0$, we find that

$$\mathrm{Tr}\left\{U^T\left[Y^{-1}+C^T\Sigma_v^{-1}C\right]^{-1}U\right\}<\gamma^2$$

If part: Inequality B2 implies that $A-MC$ is stable. This fact along with $\Sigma_v>0$ suggest that $(A,\ C\Sigma_v^{-1/2})$ are a detectable pair, guaranteeing the existence of a $Y_o>0$ that satisfies Eq. B3 (Balakrishnan, 1987). Subtracting Eq. B1 from the Inequality B2 we find

$$-(Y-Y_o)+(A-MC)(Y-Y_o)(A-MC)^T+(M-M_o)\left(CY_oC^T+\Sigma_v\right)(M-M_o)^T<0$$

Since $(A-MC)$ is stable, we find that $Y-Y_o>0\Rightarrow Y^{-1}<Y_o^{-1}\Rightarrow[Y^{-1}+C^T\Sigma_v^{-1}C]^{-1}>[Y_o^{-1}+C^T\Sigma_v^{-1}C]^{-1}$. Finally, giving

$$\mathrm{Tr}\left\{U^T\left[Y_o^{-1}+C^T\Sigma_v^{-1}C\right]^{-1}U\right\}<\gamma^2$$

Proof of Theorem 1. Only if part: The nonsingularity of A suggests that the following definition is positive definite

$$Y_o\triangleq AP_oA^T+F\Sigma_wF^T \tag{B4}$$

Substituting this into Eq. 12 we find

$$P_o=\left[Y_o^{-1}+C^T\Sigma_v^{-1}C\right]^{-1} \tag{B5}$$

$$=Y_o-Y_oC^T\left(CY_oC^T+\Sigma_v\right)^1CY_o \tag{B6}$$

Substituting these equalities into the definition of Y_o (Eq. B4) and the trace inequality, we trivially find that Eq. B1 holds and $\mathrm{Tr}\{U^T[Y_o^{-1}+C^T\Sigma_v^{-1}C]^{-1}U\}<\gamma^2$. As the assumptions of Lemma A1 hold, the existence of a $Y>0$ such that Inequality B2 holds is guaranteed. Employing the matrix inversion lemma again, we find

$$-Y+F\Sigma_wF^T+A\left[Y^{-1}+C^T\Sigma_v^{-1}C\right]^{-1}A^T<0 \tag{B7}$$

Since $Y^{-1}>0$ and A is nonsingular, we can define a positive definite matrix

$$X\triangleq(A^T)^{-1}\left[Y^{-1}+C^T\Sigma_v^{-1}C\right]A^{-1}$$

Rearranging this we find $Y=(A^TXA-C^T\Sigma_v^{-1}C)^{-1}$. Substituting these equalities into Inequality B7 we find

$$-\left(A^TXA-C^T\Sigma_v^{-1}C\right)^{-1}+F\Sigma_wF^T+X^{-1}<0 \tag{B8}$$

$$\Leftrightarrow$$

$$-A^TXA+C^T\Sigma_v^{-1}C+\left(F\Sigma_wF^T+X^{-1}\right)^{-1}>0$$

$$\Leftrightarrow$$

$$-A^TXA+C^T\Sigma_v^{-1}C+X-XF\left(F^TXF+\Sigma_w^{-1}\right)^{-1}F^TX>0$$

$$\Leftrightarrow$$

$$\begin{bmatrix}X-A^TXA+C^T\Sigma_v^{-1}C & XF\\ F^TX & F^TXF+\Sigma_w^{-1}\end{bmatrix}>0$$

And finally we have

$$\mathrm{Tr}\left\{U^T(A^TXA)^{-1}U\right\}=\mathrm{Tr}\left\{U^T\left(Y^{-1}+C^T\Sigma_vC\right)^{-1}U\right\}<\gamma^2$$

If part: Clearly, Inequality 24 is equivalent to Eq. B8. Now define $Y=(A^TXA-C^T\Sigma_v^{-1}C)^{-1}$, which is positive definite due to Eq. B8 and the fact that $X^{-1}>0$. Using this definition and Eq. B3 we easily conclude that Inequality B7 holds if A is nonsingular. Furthermore, the trace inequality requirement of Lemma A1 is trivially satisfied. Thus, by Lemma A1 there exists a $Y>0$ that satisfies Eq. B1. Finally defining $P_o\triangleq[Y_o^{-1}+C^T\Sigma_v^{-1}C]^{-1}$ and performing the necessary algebra yields the desired result.

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